Part II: Regional Residence Times and Local Flow Rates

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The analysis of Part I (Rubinovitch and Mann, 1983) is continued here, considering the movements of a single particle in an arbitrary flow system in terms of the total times it resides in various flow regions. Results from the theory of Markov chains are used to derive expressions for the joint distribution of number of visits and total residence time in a flow region and for the total regional residence time distribution. Further, the relationships between the local particle flow rate, number of visits to a flow region, and net flow rate through the system are derived. Specifically, it is shown that

$$\begin{pmatrix}\text{flow rate}\\ \text{through a}\\ \text{region}\end{pmatrix} = \begin{pmatrix}\text{net flow}\\ \text{rate through}\\ \text{the system}\end{pmatrix} \times \begin{pmatrix}\text{mean number of}\\ \text{visits to the region}\\ \text{by a fluid element}\end{pmatrix}$$

This relation is valid for any general flow system and any general region in the system. It holds true irrespective of the number of inlets and outlets to the region or of the nature of the internal mixing in the region. It is further shown how this relation leads to an experimental method for measuring local flow rates.

SCOPE

In Part I of this series (Rubinovitch and Mann, 1983) the movements of a single particle in a general flow system were described in terms of the flow regions it visits and the number of visits to various flow regions. These enabled us to characterize the trajectory of particles in the system without considering the time scale. However, in most operations the performance of the process depends on the durations of particles' visits to various regions and on the local particle flow rates. In this article, we extend the analysis and consider these and other related process characteristics.

CONCLUSIONS AND SIGNIFICANCE

In this article the movements of a particle in an arbitrary flow system are characterized in terms of the total time it resides in various regions of the system. This process characteristic together with the number of visits to various flow regions provide a means to describe mathematically the particles' history in the process. Among the results we have obtained are:

- The joint distribution of the number of visits and total residence time in a region
- The total residence time distribution in a flow region, its means and variance

These expressions are useful in analyzing processes which depend on regional residence time (e.g., conversion in high-temperature zones) and on the number of visits to a region (e.g., attrition).

Further, relationships among local flow rates, number of visits to a flow region and the overall flow rate through the system are derived. Specifically, it is shown that

$$\begin{pmatrix}\text{flow rate}\\ \text{through a}\\ \text{region}\end{pmatrix} = \begin{pmatrix}\text{net flow rate}\\ \text{through the}\\ \text{system}\end{pmatrix} \times \begin{pmatrix}\text{mean number of}\\ \text{visits to the region}\\ \text{by a fluid element}\end{pmatrix}$$

This expression together with the relation for the mean duration of a visit to the region lead to an experimental method for determining local internal particle flow rate in process units.

INTRODUCTION

In Part I we have described the movement of a particle in an arbitrary system in terms of the flow regions it visits. We considered there only the number of visits to a flow region or group of flow regions and the total number of regions visited. These parameters allowed us to characterize the trajectories of the particle in the system without considering the time scale. However, in most operations the performance of the process and product properties are related to the durations of particle's visits and to the local particle flow rates. In this paper we consider these and other process characteristics. We consider a general system like that described in Part I and derive expressions for the total regional residence time distribution in each region and its moments. Then we consider the joint distribution of the total time a particle spends in a specified region and the number of visits to that region, as well as the correlation coefficient of these two random variables. Next we consider the relationships between the number of visits to a region, their durations and the local flow rate and identify an experimental method to determine the internal flow rate in a flow region.

REGIONAL RESIDENCE TIMES

In this section we study regional residence times. We consider here the same general system shown in Figure 1 of Part I and adopt all the assumptions made there. In particular, we assume that the system is at steady state, thus all flow rates and flow patterns do not change with time. Now focus attention on one particle as it enters the system through the inlet. If this particle visits region j k times ($k = 1,2,\ldots$), we let $Y_1, Y_2, \ldots, Y_k$ denote the durations of these successive visits. We shall assume that $Y_1, Y_2, \ldots$ are independent and all have the same distribution function $H_j(y)$. By definition $H_j(y)$ is the one-pass residence time distribution in region j. Let $h_j(y)$ be the density function of $H_j(y)$ (wherever it exists), and let μ_j and σ_j^2 be the mean and variance of this distribution, respectively. Also, let T_j be the total cumulative time a particle spends in region j and $G_j(x) = P\{T_j \le x\}$ be its distribution function. Once again, when $G_j(x)$ has a density we shall denote it by $g_j(x)$. If the particle passes through region j exactly k times then T_j may be written as

$$T_j = \begin{cases} 0 & N_j = 0 \\ Y_1 + \ldots + Y_k & N_j = k, (k \ge 1). \end{cases} \tag{1}$$

Since $Y_1, Y_2, \ldots Y_{N_j}$ are conditionally independent given N_j, and all follow the same distribution function $H_j(y)$. We can immediately write the joint distribution N_j and T_j. This is

$$G_j(n,t) = P\{N_j = n,\ T_j \ge t\}$$
$$\begin{cases} P\{N_j = 0\} & n = 0 \\ P\{N_j = n\}H_j^{*n}(t) & n = 1,2,\ldots \end{cases} \tag{2}$$

where $H_j^{*n}(t)$ is the nth fold convolution of $H_j(t)$ with itself. Now using Eq. 15 of Part I and Eq. 1 above we have

$$G_j(n,t) = \begin{cases} 1 - p_j & n = 0 \\ p_j(1-q_j)q_j^{n-1}H_j^{*n}(t) & n = 1,2,\ldots, \end{cases} \tag{3}$$

where p_j is, as in Part I, the probability that a particle entering the system passes at least once through region j and q_j is the probability that the particle leaving region j will ever return to that region. From $G_j(n,t)$, we can get all the information on T_j and the joint behavior of T_j and N_j since p_j and q_j are known, or can be computed by the methods described in Part I.

We start with the computation of moments by transformation techniques. Let

$$\hat{G}_j(z,\theta) = E[z^{N_j} e^{-\theta T_j}]$$
$$= \sum_{n=0}^{\infty} \int_o^{\infty} z^n e^{-\theta t} d_t G_j(n,t),$$

for $|z| < 1$ and $\theta > 0$, be the joint transform of T_j and N_j. Then by a straightforward calculation we obtain

$$\hat{G}_j(z,\theta) = 1 - p_j + \frac{p_j(1-q_j)z\hat{H}_j(\theta)}{1 - zq_j\hat{H}_j(\theta)}. \tag{4}$$

The Laplace transform of T_j,

$$\hat{G}_j(\theta) = E[e^{-\theta T_j}] = \int_o^{\infty} e^{-\theta t} d_t G_j(t),$$

is therefore

$$\hat{G}_j(\theta) = \hat{G}_j(1,\theta) = (1 - p_j) + \frac{p_j(1-q_j)\hat{H}_j(\theta)}{1 - q_jH_j(\theta)}, \tag{5}$$

and the generating function of N_j is

$$E[z^{N_j}] = \hat{G}_j(z,0) = 1 - p_j + \frac{p_j(1-q_j)z}{1 - zq_j}. \tag{6}$$

It can be easily checked that Eq. 6 is the generating function of the distribution of N_j given by Eq. 15 of Part I. Since we already know the distribution, mean and variance of N_j (Part I), we shall have no further use of Eq. 6 here. The mean and variance of T_j can be computed easily from Eq. 5. Since

$$E[T_j] = -\frac{d}{d\theta}\hat{G}_j(\theta)\bigg|_{\theta\text{-}0}, \tag{7}$$

$$E[T_j^2] = \frac{d^2}{d\theta^2}\hat{G}_j(\theta)\bigg|_{\theta=0} \tag{8}$$

and Var $[T_j] = E[T_j^2] - E[T_j]^2$, we immediately obtain

$$E[T_j] = \frac{p_j}{1-q_j}\mu_j \tag{9}$$

$$\text{Var}[T_j] = \frac{p_j(1-p_j+q_j)}{(1-q_j)^2}\mu_j^2 + \frac{p_j}{1-q_j}\sigma_j^2. \tag{10}$$

Using Eq. 18 of Part I, the mean of T_j can also be written as

$$E[T_j] = E[N_j] \cdot \mu_j \tag{11}$$

We shall need this relation in the next section. Note that Eq. 11 can also be obtained directly from Wald's identity (Feller, 1971).

It is also interesting to note that when $p_j = 1$ and $q_j > 0$ the expressions of Eqs. 5, 6, 9 and 10 are reduced to those obtained by Mann et al. (1979) for the corresponding functions in a single continuous recycle unit with a recycle fraction q_j. Eq. 4 with $p_j = 1$ and $z = 1$ is the same as the Laplace transform of the total regional residence time distribution in the region which is on the main flow line, Eq. 6 with $p_j = 1$ is the same as the generating function of the number of passes through that region, etc. From a physical view point this means that whenever there is a region in the system through which all the material must pass (i.e., $p_j = 1$) and to which a fraction of the material does return ($q_j > 0$) this region behaves, as far as regional residence times are concerned, as if it were the unit on the main flow line of a simple continuous recycle system with recycle fraction q_j. This observation may be helpful in situations when one is interested in the total regional residence time, number of passages, etc. in only one such region. In the Example of Part I, Region 5 is a flow region with these properties.

We can now compute the covariance and correlation coefficient of N_j and T_j. For this we need the product moment of these two random variables. This can be computed using the relation

$$E[N_jT_j] = -\frac{d}{d\theta}\frac{d}{dz}\hat{G}(z,\theta)\bigg|_{\substack{\theta=0\\ z=1}},$$

however, the following argument is shorter. Using well known properties of conditional expectations, the representation of Eq. 1 above and the expression of $E[N_j^2]$ given in Eq. 19 of Part I, we can write

$$\begin{aligned} E[N_jT_j] &= E[E[N_jT_j|N_j]] \\ &= E[N_jE[T_j|N_j]] \\ &= E[N_jE[Y_1 + \ldots + Y_{N_j}|N_j]] \\ &= E[N_j^2E[Y_1]] = \mu_jE[N_j^2] \\ &= \frac{\mu_jp_j(1+q_j)}{(1-q_j)^2} \end{aligned}$$

Hence,

$$\begin{aligned} Cov[N_j,T_j] &= E[N_jT_j] - E[N_j]E[T_j] \\ &= \frac{\mu_jp_j(1+q_j)}{(1-q_j)^2} - \frac{p_j^2}{(1-q_j)^2}\mu_j \\ &= \frac{\mu_j(p_j + p_jq_j - p_j^2)}{(1-q_j)^2} \end{aligned} \tag{12}$$

It follows that the correlation coefficient of N_j and T_j is

$$\begin{aligned} \rho(N_j,T_j) &= \frac{\text{Cov}(N_j,T_j)}{\sqrt{\text{Var}[N_j]}\sqrt{\text{Var}[T_j]}} \\ &= \left[1 + \frac{p_j(1-q_j)\sigma_j^2}{(1+q_j-p_j)\mu_j^2}\right]^{-1/2} \end{aligned} \tag{13}$$

Once again, this reduces to the expression for the corresponding parameter in a recycle system when $p_j = 1$. Also note that the correlation coefficient between these two random variables is always positive (i.e., $\rho(N_j,T_j) \ge 0$). When $p_j = 0$ then $\rho(N_j,T_j) = 1$ since then $N_j = 0$ and $T_j = 0$ with probability one. When $p_j > 0$ then $0 \le \rho(N_j,T_j) \le [1 + p_j\rho_j^2/(1-p_j)\mu_j^2]^{-1/2}$ where the lower bound is at $q_j = 1$ and the higher bound at $q_j = 0$. We conclude that one could expect high correlation between N_j and T_j for small values of p_j and q_j and low correlation for large values of these parameters.

We end this section with a discussion on the distribution of T_j. The derivation of this is indeed simple but the outcome is, unlike our previous results, not in a form which can in general be easily calculated. All our previous results (on the distribution and moments of N_j, fraction of material which pass or avoid a region or set of regions, as well as the moments of T_j and $\rho(N_j,T_j)$) are in terms of known system parameters, like μ_j, σ_j^2, and the probabilities $P(i,j)$, or in terms of parameters like p_j and q_j which can be computed using the method described in Part I. Thus, numerical results can be obtained with no difficulty. Perhaps the only exception is the expression for $G_j(n,t)$ in which convolutions for $H_j(x)$ appear. However, the distribution function of T_j involves an infinite series of convolutions and this may be difficult to evaluate. In any case, the distribution function of T_j can be obtained either by inverting the Laplace transform of Eq. 5 or directly from Eq. 2, *viz*

$$G_j(t) = P\{T_j \le t\} = \sum_{n-0}^{\infty} G_j(n,t)$$

$$= (1 - p_j) + \sum_{n=1}^{\infty} p_j(1 - q_j)q_j^{n-1}H_j^{*n}(t). \quad (14)$$

In the case when $H_j(t)$ has a density $h_j(t)$, then $g_j(t)$, the density of $G_j(t)$, can be written as

$$g_j(t) = (1 - p_j)\delta(0) + \sum_{n=1}^{\infty} p_j(1 - q_j)q_j^{n-1}h_j^{*n}(t) \quad (15)$$

The last expression is also given in Gibilaro (1979) under a somewhat different setup.

LOCAL FLOW RATES

So far we have evaluated, here and in Part I, various system characteristics of interest in terms of the matrix **P** and the one-passage residence time distribution in each flow region. However, we know that flow rates and the matrix **P** are dependent on each other, and, as we shall now see, can be computed from each other. With the concept of regional residence times of the last section we can now derive some basic relations which tie together flow rates, mean number of visits to a region, mean regional residence times and the fractions p_j and q_j. The main result is Eq. 19 below which states that the flow rate through a region, say region j, is directly proportional to the net flow rate through the system, and that the proportionality coefficient is $E[N_j]$, the mean number of visits to this region.

Let j be a fixed region of a general system satisfying the assumptions of Part I. Then from Eq. 11 we have

$$E[T_j] = E[N_j]\mu_j.$$

i.e., the mean total time a particle spends in region j is equal to the mean number of visits to that region times the mean duration of these visits. We also know that the mean duration of one visit to the region is

$$\mu_j = \frac{V_j}{w_j}, \quad (16)$$

where V_j is the volume of region j, and w_j is the flow rate through that region. Thus

$$E[T_j] = \frac{E[N_j] \cdot V_j}{w_j}. \quad (17)$$

A different expression for $E[T_j]$ can be derived using the following argument. Imagine as if all flow regions of our system, except region j, are replaced by flow lines with volume zero while region j remains unchanged. This would yield a new system with interconnecting flow lines and just one flow region, region j, of volume V_j. If, for example, we choose $j = 6$ and follow this procedure for the example of Figure 2 in Part I, the system obtained is as shown in Figure 1.

It is easy to see that in the new system and in the original system the regional residence times of region j are the same (Eq. 1). Furthermore the regional residence time in region j of the new system is nothing but the total residence time in this system. As such, its mean must be equal to the volume of the system (V_j) divided by the net flow rate, w_0, through the system. In other words we have

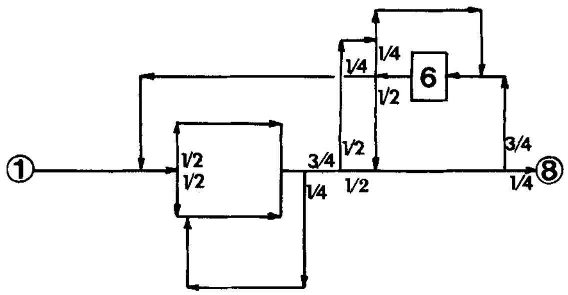

Figure 1. Example system with only Region 6.

$$E[T_j] = \frac{V_j}{w_0} \quad (18)$$

Equating now the right hand sides of Eqs. 17 and 18 leads to

$$w_j = E[N_j]w_0 \quad (19)$$

Some conceptual and practical implications of this fundamental result will be discussed later. Here, we note only that this provides a tool for computing the flow rates through each region of the system from the matrix **P**. All one has to do is to compute $E[N_j] = R(1,j)$ using the techniques of Part I and multiply it by w_0. In the example of Figure 2, Part I, we had

$$E[N_1] = 1, \quad E[N_2] = 1, \quad E[N_3] = 1.2, \quad E[N_4] = 2,$$
$$E[N_5] = 4, E[N_6] = 5.6, E[N_7] = 2.6,$$

and hence

$$w_1 = w_0, \quad w_2 = w_0, \quad w_3 = 1.2\, w_0, \quad w_4 = 2\, w_0$$
$$w_5 = 4\, w_0, \quad w_6 = 5.6\, w_0, \quad w_7 = 2.6\, w_0.$$

These flows rates can, of course, be calculated by material balances step by step on Figure 2 of Part I, using the matrix **P**. But, here we obtain all of them in one step by multiplying the vector $R(1) = (R(1,1), R(1,2), \ldots R(1,r+1))^T$ by w_0. In complex systems this may be an easier procedure.

Several additional results can be obtained from Eq. 19. First, it is interesting to consider the meaning of each term in Eq. 19. The left hand term, w_j, is the average number of particles, or fluid elements, entering (or passing through) region j in a time unit. On the right hand side $E[N_j]$ is the average number of visits to region j per particle and w_0 is the average number of particles entering the system in a time unit. With this we can rewrite Eq. 19 as

$$\left(\frac{\text{mean number of enteries to region } j}{\text{time}}\right) = \left(\frac{\text{mean number of entries to region } j}{\text{mean number of entries to the system}}\right) \times \left(\frac{\text{mean number of entries to the system}}{\text{time}}\right)$$

This expression is similar to a material balance statement, and it ties the results obtained by the probabilistic approach to those obtained by the deterministic approach. Such relationships will be discussed in detail in subsequent parts of this series.

Another result can be obtained if we substitute $E[N_j]$ in Eq. 19 according to Eq. 18 of part I. This leads to

$$w_j = \frac{p_j}{1 - q_j} w_0. \quad (20)$$

This result may be useful in situations when one can determine with ease, (by *ad hoc* calculation), any two of w_j, p_j and q_j. Then the third parameter can be evaluated using Eq. 20 and one can immediately obtain various results on parameters like $E[T_j]$ and $E[N_j]$ without having to go through the computations of matrix **R**.

It is important to realize that the results of Eqs. 19 and 20 are not, in general, equivalent. As we shall see in the next section Eq. 19 is true for completely general systems with flow regions which may have several inlets and several outlets. On the other hand, Eq. 20 is true only for the case when each flow region has one inlet and one outlet, or when the flow region is well mixed. Indeed, Eq. 20 was first discovered by Gibilaro (1979) under these assumptions using different methods.

REGIONS WITH SEVERAL INLETS AND OUTLETS

In this section we show that Eq. 19 holds true for any arbitrary flow region of volume V_j and flow rate w_j where the latter is the total inlet (or outlet) flow rate passing through the boundaries of the region. Such a general zone can be described by a flow region with several inlets and several outlets. We shall not present here a complete analysis of such systems (this will be done elsewhere using more advanced probabilistic techniques) but will only show that the basic result of Eq. 19 holds true for any flow region.

Suppose that we are given a general system as in part I except that we no longer require that each flow region has only one inlet and one outlet. Then, if region j is not well mixed the probability that a particle which enters it from one inlet, will exit through a specified outlet may be different from the probability that a particle which enters from another inlet will exit from the same outlet. Consequently, it is not hard to see that Eq. 1 of part I is no longer true, that the process X is not necessarily a Markov chain and the previous analysis does not apply. Our objective now is to show that in spite of this, Eq. 19 holds true.

Let us focus attention on a specified region, say region j. If region j has several inlets and just one outlet there is no problem since then one can lump together all the inlets, the process X is a Markov chain and the previous analysis applies. Similarly, if region j has several outlets and one inlet, the outlets can be lumped together. So, consider the case when region j has α inlets and several outlets. In this case we change region j as follows. Instead of region j we introduce α new regions $(j_1, \ldots, j_\alpha)$ with inlet 1 leading to region j_1 only, inlet 2 leading to region j_2 only, etc., as shown for example in Figure 2. Each of the new regions will have all the outlets as the original region j.

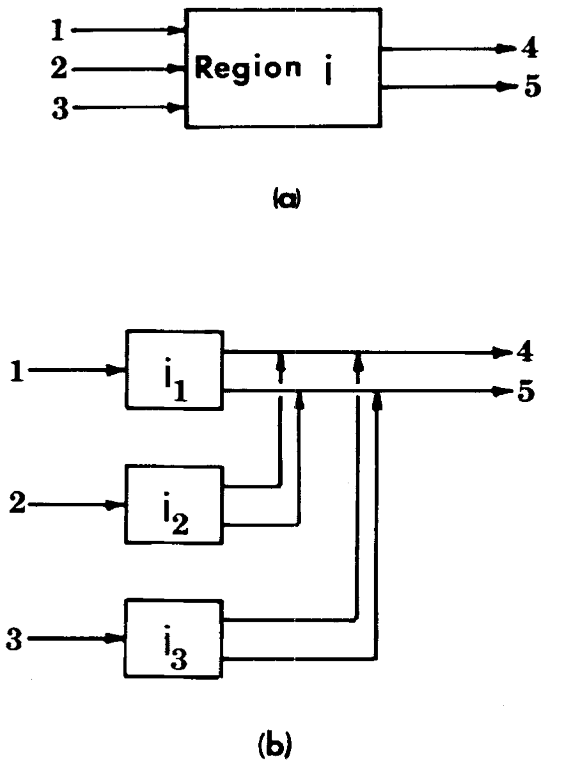

Figure 2. Decomposition of a flow region with several inlets and several outlets.

If we repeat this procedure for all regions of the system we obtain a new system in which each flow region has just one inlet and one outlet. The previous analysis applies to the new system and hence, for each flow region in the new system (Eq. 19) is true. In particular, for regions $j_1, \ldots j_\alpha$ we have

$$w_{j_1} = E[N_{j_1}]w_0 \quad \vdots \quad w_{j_\alpha} = E[N_{j_\alpha}]w_0. \tag{21}$$

Noting now that the number of visits to region j, N_j, is the same as $N_{j_1} + \ldots + N_{j_\alpha}$ we conclude that

$$E[N_j] = E[N_{j_1}] + \ldots + E[N_{j_\alpha}],$$

and hence, summing all the equations of 21 we obtain

$$w_j = E[N_j]w_0.$$

Thus, in completely general systems consisted of flow regions with several inlets and several outlets, Eq. 19 is true with w_j being the sum of the flow rates of all lines leading to region j.

As already mentioned, we have only proven that Eq. 19 is true for this general system while the rest of our analysis does not apply. However, the method of decomposing flow regions, just discussed, provides also the means for computing some other process characteristics of interest. For example, the propability p_j of ever reaching region j can be calculated by first decomposing region j into α regions each having one inlet and one outlet and then using the technique described in the last section of Part I to compute the probability of never visiting any of the regions in the set $A = j_1, \ldots j_\alpha$. The same applies to the mean number of visits to region j. It does not apply however to the computations of the variance of N_j since $N_{j_1}, \ldots, N_{j_\alpha}$ are not necessarily independent.

AN EXAMPLE

To illustrate how a region with several inlets and outlets is decomposed, consider the system shown schematically in Figure 3. First, consider a system consisting of three well-mixed flow regions with backflow connected in series shown in Figure 3a. This system is commonly used as a flow model to describe deviations from plug flow (see for example, Klinkenber, 1966). The forward and backward flow rates are indicated in the figure. By Eq. 19 the expected number of visits to the various flow regions are:

$$E[N_2] = \frac{w_o + a}{w_o}$$

$$E[N_3] = \frac{w_o + 2a}{w_o}$$

$$E[N_4] = \frac{w_o + a}{w_o}$$

Incidentally, the same system with more than three flow regions

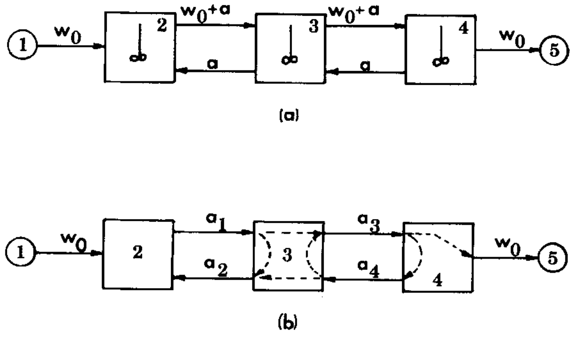

Figure 3. Cascade of flow regions with backflow.

will have similar mean number of visits to each region. The mean number of visits to the end regions is $(w_o + a)/w_o$ and to all other regions is $(w_o + 2a)/w_o$.

Now let us abandon the assumption that all regions are well mixed. Instead, we shall assume that in regions with several outlets (Regions 3 and 4) particles entering from the left have a different likelihood of moving to the left than particles which enter from the right. Note that this is a completely different case than the one considered in the previous paragraph. When a flow region is well mixed, the probability of a particle exiting the region through a certain outlet is equal to the fractional outlet flow rate through that outlet, and is independent of the inlet through which it enters the region. When a flow region is not well mixed, an outlet stream may consist of more particles which enter from one inlet than from another. To describe the movement of a particle (and also to express the composition of the various outlet streams), one should express these probabilities explicitly. In the example here, we shall assume that particles which enter flow Region 3 or 4 from the left have probability β of exiting from the left and $1-\beta$ of exiting from the right exit. Similarly, in Region 4 a particle which enters from the left (the only entrance) will with probability β return to Region 3 and with probability $1-\beta$ will exit the system. These probabilities are shown in Figure 3b. In this case the flow rates are not the same as those indicated in Figure 3a.

To verify that Eq. 19 holds true for this case we shall compute $E[N_2]$, $E[N_3]$ and $E[N_4]$ using two methods. First, using Eq. 19 directly and then using the decomposition method described above. To compute these by Eq. 19 we start by calculating the various flow rates in the system according to the material balance equations

$$a_1 = w_o + a_2$$

$$a_1 + a_4 = a_2 + a_3$$

and the two probability relations

$$a_2 = \beta a_1 + (1-\beta)a_4$$

$$a_4 = \beta a_3.$$

The calculated flow rates are thus:

$$a_1 = \frac{w_o(1+\beta)}{1-\beta} \qquad a_2 = \frac{2\beta w_o}{1-\beta}$$

$$a_3 = \frac{w_o}{1-\beta} \qquad a_4 = \frac{\beta w_o}{1-\beta}$$

and the net flow rates through the individual flow regions are

$$w_2 = w_o + a_2 = \frac{w_o(1+\beta)}{1-\beta}$$

$$w_3 = a_1 + a_4 = \frac{w_o(1+2\beta)}{1-\beta}$$

$$w_4 = a_3 = \frac{w_o}{1-\beta}$$

Hence, by Eq. 19 the respective means number of visits are

$$E[N_2] = \frac{1+\beta}{1-\beta}$$

$$E[N_3] = \frac{1+2\beta}{1-\beta} \tag{22}$$

$$E[N_4] = \frac{1}{1-\beta}$$

Now let us compute $E[N_2]$, $E[N_3]$ and $E[N_4]$ using the decomposition technique and the method described in Part I. In the decomposition we have to replace Region 3 by two regions, 3I and 3II, each with one inlet. Nothing has to be changed in Regions 2 and 4 since the former has only one outlet and the latter only one inlet. The new decomposed system is shown in Figure 4. The transition probabilities are indicated on the figure and each region has now only one inlet, or alternatively, just one outlet. The matrix of transition probabilities for this system is (the rows and columns correspond to flow Regions 1, 2, 3I, 3II, 4 and 5 in this order)

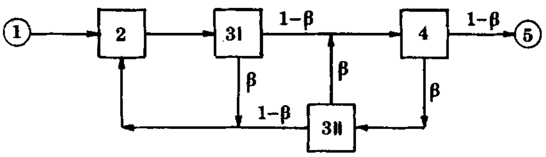

Figure 4. Decomposition of Region 3.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 1-\beta & 0 \\ 0 & 1-\beta & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta & 0 & 1-\beta \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Note that in this case all particles move from Region 1 to Region 2 so we can simplify the computation by considering Region 2 as the origin, delete region 1 from the system and work with the matrix obtained from Eq. 23 above by deleting the first row and first column.

$$\bar{\bar{P}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \beta & 0 & 0 & 1-\beta & 0 \\ 1-\beta & 0 & 0 & \beta & 0 \\ 0 & 0 & \beta & 0 & 1-\beta \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{24}$$

(In Eq. 24, row and columns correspond to flow Regions 2, 3I, 3II, 4 and 5 in this order.) Now, using the procedure described in Part I, we have

$$\bar{\bar{Q}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \beta & 0 & 0 & 1-\beta \\ 1-\beta & 0 & 0 & \beta \\ 0 & 0 & \beta & 0 \end{bmatrix},$$

and

$$\bar{\bar{R}} = (\bar{\bar{I}} - \bar{\bar{Q}})^{-1} = \begin{bmatrix} \frac{1+\beta}{1-\beta} & \frac{1+\beta}{1-\beta} & \frac{\beta}{1-\beta} & \frac{1}{1-\beta} \\ \frac{\beta}{1-\beta} & \frac{1+\beta}{1-\beta} & \frac{\beta}{1-\beta} & \frac{1}{1-\beta} \\ \frac{1}{1-\beta} & \frac{1}{1-\beta} & \frac{1}{1-\beta} & \frac{1}{1-\beta} \\ \frac{\beta}{1-\beta} & \frac{\beta}{1-\beta} & \frac{\beta}{1-\beta} & \frac{1}{1-\beta} \end{bmatrix}$$

From this

$$E[N_2] = R(1,1) = \frac{1+\beta}{1-\beta}$$

$$E[N_3] = E[N_{3I}] + E[N_{3II}] = R(1,2) + R(1,3) = \frac{1+2\beta}{1-\beta}$$

$$E[N_4] = R(1,4) = \frac{1}{1-\beta}$$

These values are in agreement with the expected number of visits calculated from flow rates in Eq. 22.

APPLICATIONS

The applications of the foregoing results can be divided into two categories: those related to the total regional residence times, and those related to local flow rates. For processes which depend on the total regional times (for example, reaction in a high temperature zone, agglomeration, etc.), Eqs. 14 and 15 provides the mathematical means to express property deviations among individual particles. For processes which depend on two operating parameters such as the total regional time and the number of visits to a flow region (e.g., reaction in a high temperature zone and attrition) the joint distribution provides the mathematical means to express the deviations among individual particles. The implementation of these results depends on developing relationships between the changes the particles undergo and these characteristic random variables. We believe that with the awareness of these results, such relationships will be investigated and developed.

Perhaps the most important practical result of the analysts is that Eq. 19 indicates an experimental method for determining local particle flow rate in a process. The method is based on measuring the number of times a single tagged particle (e.g., radioactive particle) visits a certain flow region as it passes through the system. This is done by introducing a tagged particle in the inlet of the system and then counting the number of times it visits a certain flow region before appearing in the outlet line of the system. If this is repeated several times the average number of visits, $E[N_j]$, can be easily determined. Then, if the net flow rate through the system, w_o, is known, the local flow rate, w_j, can be computed from Eq. 19. Various considerations on the implementation of this technique and an extension to batch systems are discussed elsewhere (Mann and Rubinovitch, 1983).

CONCLUDING REMARKS

In this part of the series we have considered the relations between number of visits to a flow region, total regional residence time and local flow rates. We have shown how these relations can be used to determine experimentally the local particle flow rate. It is important to note that although these results were derived for particulate systems they also apply for homogeneous fluid. Whether a fluid or particle system is considered, the results illustrate how the approach based on describing the movement of a single particle and using probabilistic techniques lead to new and useful results. In the next part of the series we shall extend the analysis to systems with several inlets and several outlets.

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NOTATION

$G_j(t)$ = distribution function of total residence time in region j
$g_j(t)$ = density function of $G_j(t)$
$G_j(n,t)$ = joint distribution of N_j and T_j (Eq. 2)
$H_j(y)$ = distribution function of the duration of one visit to region j
$h_j(y)$ = density function of $H_j(y)$
N_j = number of times a particle visits region j
$\mathbf{P}$ = matrix of transition probabilities (Eq. 4 of Part I)
p_j = probability of a particle ever visiting region j
$\mathbf{Q}$ = matrix of transition probability from a transient state to a transient state (Eq. 6 of Part I)
q_j = probability of a particle leaving region j ever returning to that region
$R(i,j)$ = expected number of visits to flow region j by a particle leaving flow region i
$\mathbf{R}$ = a matrix whose entries are $R(i,j)$
T_j = total residence time in region j, random variable (Eq. 1)
t = time
V_o = total volume of the system
V_j = volume of flow region j
w_o = net flow rate through the system, volume/time
w_j = net flow rate through region j
X = Markov chain (Eq. 1 of Part I)
x = time
Y_i = random variable denoting the duration of the ith visit to a flow region, time
y = time
z = transform variable, dimensionless

Greek Letters

α = number of subregions each with one inlet or one outlet
β = fraction parameter
$\delta(o)$ = delta dirac function at the origin
θ = transform variable
μ_j = mean of $H_j(y)$, time
σ_j^2 = variance of $H_j(y)$, time^2

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